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ANALYSIS & DESIGN OF FRACTANCE BASED FRACTIONAL ORDER FILTER

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Abstract Analysis of analog filter of fractional order is the main objective of this paper. The basics of fractance device are studied. Rational approximation of fractional order operator using continued fraction expansion is carried out. Analytical expressions of the differentiator operator for different input signals have been developed and simulated in MATLAB software and corresponding simulation results are shown for different orders of differentiation. Fractional order filter is studied and performance of the fractional order filter is checked for different input signals (sine wave, trapezoidal wave, sawtooth wave and chirp signal) with random noise and simulated in MATLAB software and the resulted output is compared with the integer order filter output.

Keywords Fractional Calculus, Continued Fraction Expansion, Fractional Order Operator, Fractance Device.

I. INTRODUCTION

In recent years it has turned out that many phenomena in engineering, physics, chemistry, and other sciences can be described very successfully by models using mathematical tools from fractional calculus [1]. To obtain better performance, in last few decades, several applications based on fractional order modeling in wide spread fields of science and engineering have been proposed. This includes fluid flow, optics, geology, behavior of visco-elastic material, bioscience, medicine, non-linear control, signal processing, etc [2].

Fractional calculus is three centuries old as the conventional calculus. All of us are familiar with normal derivatives and integrals, like, df/dt, d^2f/dt^2, $\int_0^t f(u)\,du$. We have first-order, second order derivatives, or first integral, double integral, of a function. Now we wish to have half order, π^{th} order, or derivative of a function. So, fractional calculus is equal to derivatives and integrals of arbitrary real or complex order. Fractional-order calculus is an area of mathematics that deals with derivatives and integrals from non-integer orders. In other words, it is a generalization of the traditional calculus that leads to similar concepts and tools, but with a much wider applicability. fundamental operator ${}_aD^{\alpha}_t$ $\alpha \in R$, where a and t are the limits and α is the order of the operation [3].

$$ {}_aD^{\alpha}_t = \begin{cases} \frac{d^{\alpha}}{dt^{\alpha}}, & \alpha > 0 \\ 1, & \alpha = 0 \\ \int_a^t (d\tau)^{-\alpha} & \alpha < 0 \end{cases} \tag{1} $$

Some special functions used in fractional calculus are Gamma Function, Beta Function, and Mittag-Leffler Function [4].

The paper is organized as follows: In section-2, the study of fractance device is presented. In section-3 rational approximation of fractional order operator using different methods (Newton, Mastuda, Oustaloup and CFE method) is presented and compared with the ideal response. Analytical expressions of the differentiator operator for different input signals have developed and corresponding simulation results have been shown for different orders of differentiation. In section-4 fractional order filter is studied and performance of the fractional order filter is checked for different input signals (sine wave, trapezoidal wave, sawtooth wave and chirp signal) with random noise and the resulted output is compared with the integer order filter output.Section-5 presents conclusion.

1.1 Fractional Differ Integral Definitions

1.1.1 Grunwald-Letnikov, Riemann-Liouville and Caputo Definitions

There are two main approaches for defining a fractional derivative. The first considers differentiation and integration as limits of finite differences. The Grunwald-Letnikov definition follows this approach. The other approach generalizes a convolution type representation of repeated integration. The Riemann-Liouville and Caputo definitions take this approach. Riemann-Liouville and Caputo fractional derivatives are fundamentally related to fractional integration operators. Consequently, the initial conditions of fractional derivatives are the frequency distributed and infinite dimensional state vector of fractional integrators.

1.1.2 Grunwald-Letnikov Definition

The Grunwald-Letnikov approach presents limit definitions for higher order derivatives and integrals and shown that [5]

$$_aD^\alpha_t f(t) = lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k\to 0}^{\frac{t}{h}}(-1)^k\binom{\alpha}{k}f(t-\alpha h) \qquad (2)$$

where h is the time increment.

1.1.3 Riemann-Louville Definition

Riemann-Louville approach generalizes a convolution type representation of repeated integration and given by [5]

$$_aD^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)}(d/dt)^n\int_a^t (f(\tau)\,d\tau)/(t-\tau)^{\alpha-n+1},$$

$$(n-1 \le \alpha < n) \qquad (3)$$

where Γ is the Euler's gamma function.

1.1.4 Caputo Fractional Derivative

The Caputo definition of fractional differentiation of fractional order α, can be written as [5]

$$D^\alpha_* f(t) = J^{m-\alpha}D^m f(t)$$

$$with \quad m-1<\alpha\le m, \qquad (4)$$

$$D^\alpha_* f(t)=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^t f^m(\tau),\ m-1<\alpha<m\\ \frac{d^m}{dt^m}f(t), \alpha=m\end{cases}$$

where Γ is the Euler's Gamma function.

II. FRACTANCE DEVICE

Fractance device is an electrical element which exhibits fractional order impedance properties. The impedance of the fractance device is defined as.

$$Z(j\omega) = (j\omega)^\alpha \qquad (5)$$

where ω is the angular frequency and α takes the values as −1, 0, 1 for capacitance, resistance, and the inductance, respectively. Fractance device finds applications in robotics, hard disk drives, signal processing circuits, fractional order control, and so forth [6].

The *Curie law* is as follows. Suppose that the voltage $v(t) = Uu(t)$ is applied to a capacitor possessing no initially stored charged. That is, there is no energy stored in the device before applying the DC voltage U. The current through the device will have the general form [7]

$$i(t) = \frac{U}{ht^\alpha} \quad for\ t>0 \text{ and } 0<\alpha<1 \qquad (6)$$

(h and U are real-valued). This is a power law dependence of terminal current upon the input voltage. The Laplace transform of the input voltage is

$$v(s) = \frac{U}{s} \qquad (7)$$

However, the Laplace transform of $i(t)$ is

$$i(s) = \frac{\Gamma(1-\alpha)U}{hs^{(1-\alpha)}} \qquad (8)$$

Here $\Gamma(x)$ is the gamma function. Normally, the impedance of a two-terminal linear time-invariant (LTI) circuit element is defined to be

$$z(s) = \frac{v(s)}{i(s)} \qquad (9)$$

For the Curie law device of (6) from (9) it can be seen

$$z(s) = \frac{h}{\Gamma(1-\alpha)}\frac{1}{s^\alpha} \qquad (10)$$

In (10) $0 < \alpha < 1$ and so (10) is a considered a fractional impedance, or fractance for short.

The following are some of the important points about fractance device:

(i) The phase angle is constant with frequency but depends only on the value of fractional order, α. Hence this device is also called as constant phase angle device or simply *fractor.*
(ii) Moderate characteristics between inductor, resistor, and capacitor can be obtained using fractance device.
(iii) By making use of an operational amplifier, a fractional order differentiation and integration can be accomplished easily.

Design of fractance having given order α can be done easily using any of the rational approximations or a truncated continued fraction expansion (CFE), which also gives a rational approximation. Truncated CFE does not require any further transformation; a rational approximation based on any other methods must be first transformed to the form of a continued fraction; then the values of the electrical elements, which are necessary for building a fractance, are determined from the obtained finite continued fraction.

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III. RESULTS AND DISCUSSION

3.1 Implementation of Fractional Order Differentiator Operator $s^{0.5}$

The output of fractional order differentiator is such that the order of the differentiation of input signal may be either real or complex i.e. the power of the *s* (in the Laplace domain) is arbitrary instead of integer only as in case of conventional differentiator. The fractional order differentiators or integrators are defined, in the Laplace domain, by the following transfer function

$$H(s)=s^{\alpha} \qquad (11)$$

where *s* is the Laplace operator.

By making use of well known Regular Newton Process, Carlson and Halijak have obtained rational approximation of $1/\sqrt{s}$ as[8]

$$H(s)=\frac{s^4+36s^3+126s^2+84s+9}{9s^4+84s^3+126s^2+36s+1} \qquad (12)$$

By approximating an irrational function with rational one, and fitting the original function in a set of logarithmically spaced points, Mastuda has obtained rational approximation of $1/\sqrt{s}$ [9]

$$H(s)=\frac{0.08549s^4+4.877s^3+20.84s^2+12.995s+1}{s^4+13s^3+20.84s^2+4.876s+0.08551} \qquad (13)$$

Oustaloup has approximated the fractional differentiator operator s^{α} by a rational function and derived the following approximations [10]

$$1/\sqrt{s}=\frac{s^5+74.97s^4+768.5s^3+1218s^2+298.5s+10}{10s^5+298.5s^4+1218s^3+768.5s^2+74.97s+1} \qquad (14)$$

$$\sqrt{s}=\frac{10s^5+298.5s^4+1218s^3+768.5s^2+74.97s+1}{s^5+74.97s^4+768.5s^3+1218s^2+298.5s+10} \qquad (15)$$

3.2 Continued Fraction Expansion

It is known that the continued fraction expansion for $(1+x)^{\alpha}$ as [11]

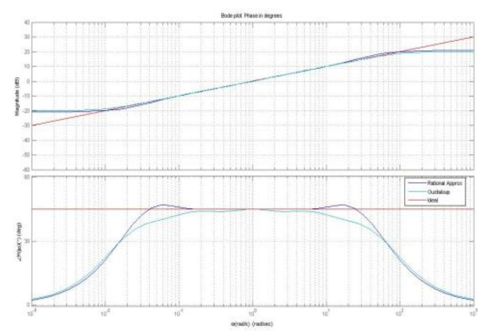
$$(1+x)^{\alpha}=\frac{1}{1-}\,\frac{\alpha x}{1+}\,\frac{(1+\alpha)x}{2+}\,\frac{(1-\alpha)x}{3+}\,\frac{(2+\alpha)x}{2+}\,\frac{(2-\alpha)x}{5+-----} \qquad (16)$$

The above continued fraction expansion converges in the finite complex s-plane, along the negative real axis from $x=-\infty$ to $x=-1$. Substituting $x=s-1$ and taking number of terms of equation, the calculated rational approximations for $\sqrt{s}$ are presented in Table 1.

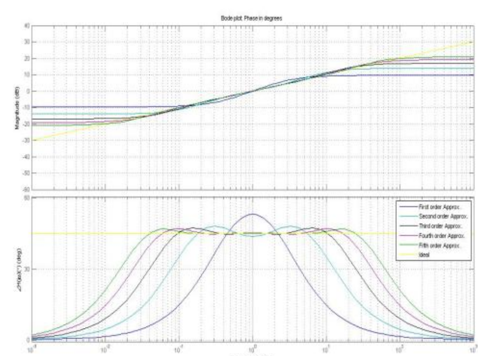
Figures 1(a) and 1(b) compare the magnitude and phase responses of the rational approximations with the ideal one [12].

TABLE 1: Rational approximations for $s^{0.5}$

S.No	No. of terms	Rational approximation
1	2	$\frac{3s+1}{s+3}$
2	4	$\frac{5s^2+10s+1}{s^2+10s+5}$
3	6	$\frac{7s^3+35s^2+21s+1}{s^3+21s^2+35s+7}$
4	8	$\frac{9s^4+84s^3+126s^2+36s+1}{s^4+36s^3+126s^2+84s+9}$
5	10	$\frac{11s^5+165s^4+462s^3+330s^2+55s+1}{s^5+55s^4+330s^3+462s^2+165s+11}$



(a)



(b)

Figure1. Comparison of magnitude and phase responses of rational approximation functions with ideal $\sqrt{s}$.

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ISSN 2321 – 5526

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Vol. 1, Issue 3, June 2013

3.3 Rational Approximations for s^α

Fractional order systems are systems that are described by fractional differential equations in which the integer order n of the derivative operator $D^n=\frac{d^n}{dt^n}$ is generalized to real or complex order α, such that one can define the operator [13]

$$D^\alpha=\frac{d^\alpha}{dt^\alpha} \qquad (17)$$

Among existing fractional systems, we find the fractance device, $PI^\lambda D^\beta$ controller, fractional order differentiators or integrators. The fractional order differentiators or integrators are defined, in the Laplace domain, by the following transfer function

$$H(s)=s^\alpha \qquad (18)$$

where s is the Laplace operator. These systems are used to calculate the fractional order time derivative and integral of an input signal. They find applications in many fields of science and engineering particularly in control and signal processing. However, such systems have Unlimited memory, thus they cannot be implemented exactly. Many algorithms have been developed to best approximate the fractional order operator s^α with analogue or digital integer models. Fractional order elements are the building blocks for the fractional order system theory, control and signal processing. The only problem with fractional order elements is its hardware realization due to its infinite dimensional nature. In practice, factional order elements can be approximated as higher order rational transfer functions which have a constant phase curve within a certain frequency band. The fractional order elements can be rationalized as analog filters by various iterative techniques like Carlson's, Oustaloup's, Charef's and CFE (continued fraction expansion) method etc. The rational approximations obtained for s^α represented in Table 2. The following plots from Figures 2(a) to 2(g) compare the magnitude and phase responses for s^α obtained using CFE method for different values of α.

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ISSN 2321 – 5526

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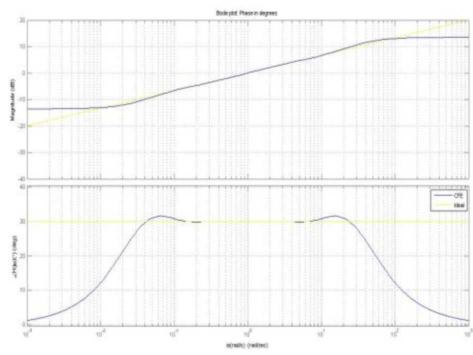
Vol. 1, Issue 3, June 2013

TABLE 2: Rational approximations for s^{α} using continued fraction expansion

No. of terms	Rational Approximation
2	$\frac{(1-\alpha)+s(1+\alpha)}{(1+\alpha)+s(1-\alpha)}$
4	$\frac{(\alpha^2+3\alpha+2)s^2+(8-2\alpha^2)s+(\alpha^2-3\alpha+2)}{(\alpha^2-3\alpha+2)s^2+(8-2\alpha^2)s+(\alpha^2+3\alpha+2)}$
6	$\frac{(\alpha^3+6\alpha^2+11\alpha+6)s^3+(-3\alpha^3-6\alpha^2+27\alpha+54)s^2+(3\alpha^3-6\alpha^2+27\alpha+54)s+(-\alpha^3+6\alpha^2-11\alpha+6)}{(-\alpha^3+6\alpha^2-11\alpha+6)s^3+(3\alpha^3-6\alpha^2+27\alpha+54)s^2+(-3\alpha^3-6\alpha^2+27\alpha+54)s+(\alpha^3+6\alpha^2+11\alpha+6)}$
8	$\frac{p0s^4+p1s^3+p2s^2+p3s+p4}{q0s^4+q1s^3+q2s^2+q3s+q5}$ where p0= q4= α^4+10 α^3+35 α^2+50 α+24 p1= q3= -4α^4-10 α^3+40 α^2+320 α+384 p2= q2= 6α^4-1 50 α^2+864 p3= q1= -4α^4+20 α^3+40 α^2-320 α+384 p4= q0= α^4-10 α^3+35 α^2-50 α+24
10	$\frac{p0s^5+p1s^4+p2s^3+p3s^2+p4s+p5}{q0s^5+q1s^4+q2s^3+q3s^2+q4s+q5}$ where p0= q5= - $\alpha^5-15\alpha^4$-85 α^3-225 α^2-274 α-120 p1= q4= 5$\alpha^5+45\alpha^4$+5 α^3-1005 α^2-3250 α-3000 p2= q3= -10α^5-30α^4+410 α^3+1230 α^2-4000 α-12000 p3= q2= 10α^5-30α^4-410 α^3+1230 α^2+4000 α-12000 p4= q1= - 5$\alpha^5+45\alpha^4$-5 α^3-1005 α^2+3250 α-3000 p5= q0= $\alpha^5-15\alpha^4$+85 α^3-225 α^2+274



(a)

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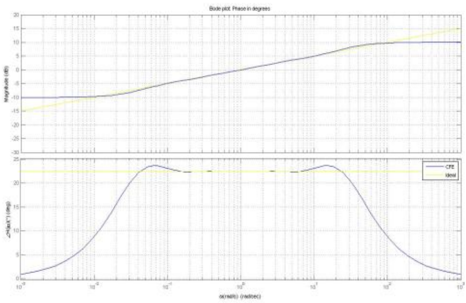
Vol. 1, Issue 3, June 2013



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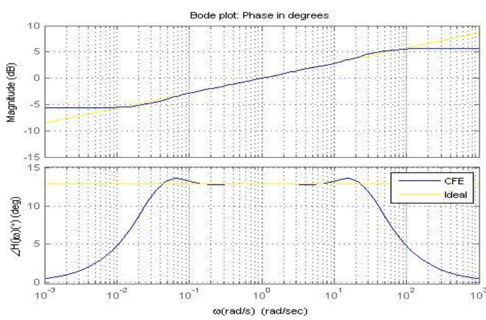

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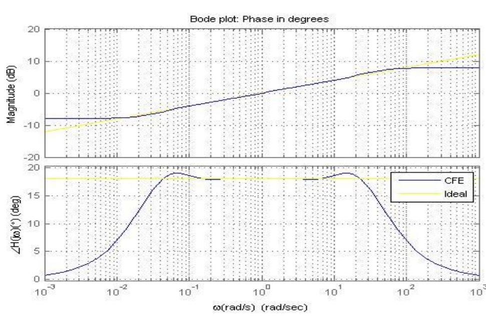

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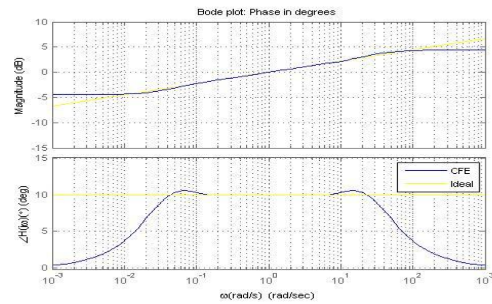

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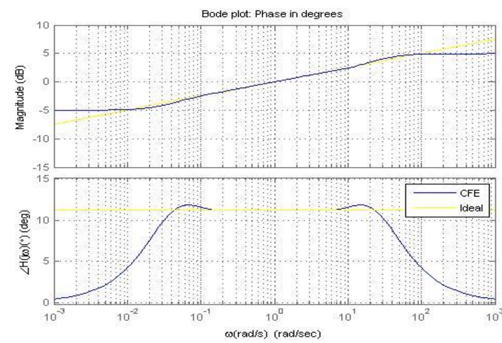

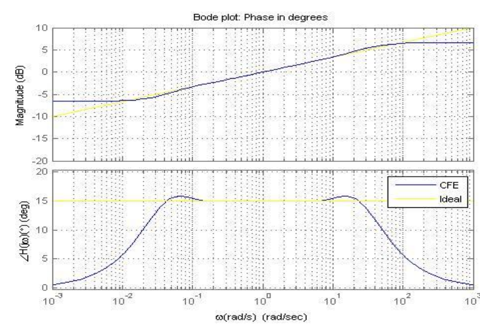

(e)

(g)

Figure 2. Comparison of magnitude and phase responses of rational approximation functions with ideal s^{α}, where α is order of the operator for figure (a) to (g) α = 0.2, 0.3 0.4, 0.6, 0.7, 0.8, and 0.9 respectively.

3.4 Time-Domain Response of s^{α} Operator

The time domain response of s^{α} differentiator operator is simulated in MATLAB. The performance of the fractional order operator is checked by giving different input signal (impulse, step, sine, and cosine input) as input of fractional order operator and for different values of α (0.1 to 0.9).

ISSN 2321 – 2004
ISSN 2321 – 5526

INTERNATIONAL JOURNAL OF INNOVATIVE RESEARCH IN ELECTRICAL, ELECTRONICS, INSTRUMENTATION AND CONTROL ENGINEERING
Vol. 1, Issue 3, June 2013

3.4.1 Response with impulse input

$$i(t) = \delta(t) \longrightarrow \boxed{s^{\alpha}} \longrightarrow p(t)$$

Let the input to the fractional order operator s^{α}, $i(t) = \delta(t)$ [14]
In the Laplace domain

$I(s) = 1$,
Output in the Laplace domain, $P(s)$ and time domain response of above system can be written as

$$p(t) = L^{-1}(I(s)*s^{\alpha})$$

$$p(t) = \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} \qquad (19)$$

The response of s^{α} with impulse input is shown in Figure3 for different values of α.

3.4.2 Response with unit step input

$$i(t)=u(t \longrightarrow \boxed{s^{\alpha}} \longrightarrow p(t)$$

Let the input to the fractional order operator s^{α}, $i(t) = u(t)$
In the Laplace domain

$I(s) =\frac{1}{s}$,

Output in the Laplace domain, $P(s)$ and time domain response of above system can be written as

$$p(t) = L^{-1}(I(s)*s^{\alpha})$$

$$p(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \qquad (20)$$

The response of s^{α} with unit step input is shown in Figure4 for different values of α.

3.4.3 Response with sine input

$$i(t) =A\sin\omega t \longrightarrow \boxed{s^{\alpha}} \longrightarrow p(t)$$

Let the input to the fractional order operator s^{α},

$i(t) =A\sin\omega t$
In the Laplace domain

$I(s) = \frac{\omega}{s^2+\omega^2}$,

Were $A=1$, amplitude of sine wave.
Time domain response of above system can be written

$$p(t) = L^{-1}(I(s)*s^{\alpha})$$

$$p(t)=A\omega*t^{1-\alpha}E*_{2,2-\alpha}(\omega^2*t^2) \qquad (21)$$

The response of s^{α} with sine wave is shown in Figure5 for different values of α.

3.4.4 Response with cosine Input

$$i(t)=A\cos\omega \longrightarrow \boxed{s^{\alpha}} \longrightarrow p(t)$$

Let the input to the fractional order operator s^{α}, $i(t)=A\cos\omega t$
In the Laplace domain

$I(s) =\frac{s}{s^2+\omega^2}$,

Were $A=1$, amplitude of cosine wave.
Output in the Laplace domain, $P(s)$ and time domain response of above system can be written as

$$p(t) = L^{-1}(I(s)*s^{\alpha})$$

$$p(t)=A\omega*t^{-\alpha}E*_{2,1-\alpha}(-\omega^2*t^2) \qquad (22)$$

The response of s^{α} with cosine wave is shown in Figure6 for different values of α.

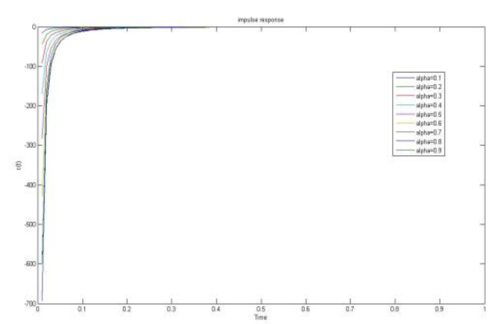

Figure3. Response of the fractional order differentiator operator for impulse input and different values of α (0.1 to 0.9). (α is the order of fractional order differentiator operator)

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Vol. 1, Issue 3, June 2013

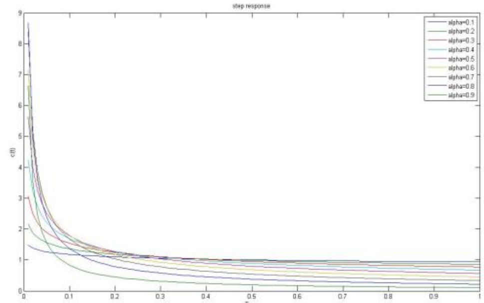

Figure4. Response of the fractional order differentiator operator for step input and different values of α (0.1 to 0.9). (α is the order of Fractional order differentiator operator).

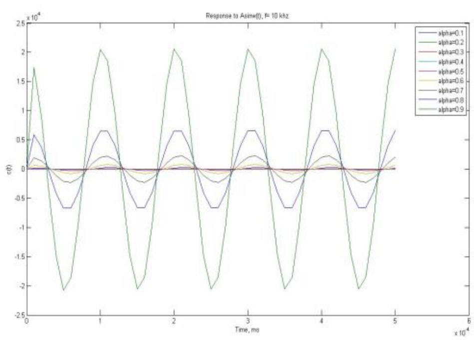

Figure5. Response of the fractional order differentiator operator for sine input and different values of α (0.1 to 0.9). (α is the order of fractional order differentiator operator)

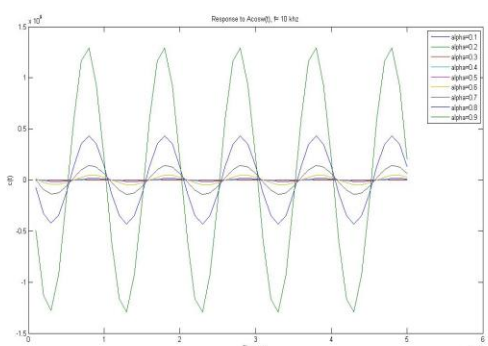

Figure 6. Response of the fractional order differentiator operator for cosine input and different values of α (0.1 to 0.9). (α is the order of fractional order differentiator operator)

3.5. Fractional Order Filter

Traditional continuous-time filters are of integer order. However, using fractional calculus, filters may also be represented by the more general fractional-order differential equations in which case integer-order filters are only a tight subset of fractional order filters. In this work, we will show that low-pass filters can be realized with circuits incorporating a single fractance device. For designing passive or active filter, filters necessarily incorporate inductors and capacitors, the total number of inductor or capacitor dictates the filter order. However, an inductor or capacitor is not but a special case of the more general so-called fractance device; which is an electrical element whose impedance in the complex frequency domain is given by $Z(j\omega) = (j\omega)^{\alpha}$ For the special case of $\alpha = 1$ this element represents an inductor while for $\alpha = -1$ it represents a capacitor[15]. Figures7 (a) and 1(b) shows the integer order and fractional order filter respectively.

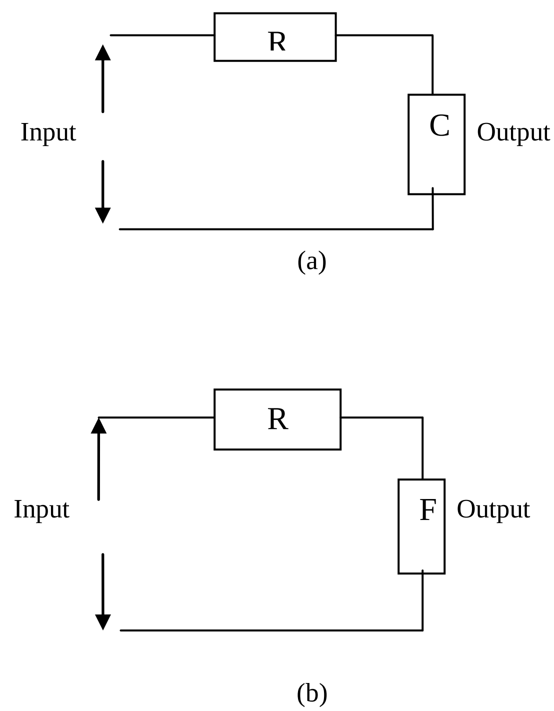

Figure7. (a) low pass filter with integer order, (b) low pass filter with fractance device

3.6 Comparison of Integer Order and Fractional Order Filter Performance

The performance of fractional order filter simulated in MATLAB is checked by using Sine, sawtooth wave, trapezoidal wave with random noise as input and resulted output compared with the output of the integer order filter with same input. And for the optimization, the performance of the fractional order filter is checked for different values of α and simulation is done in MATLAB.

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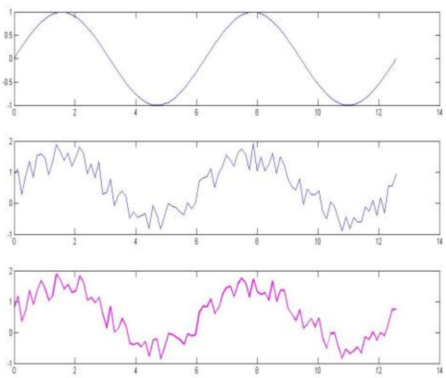
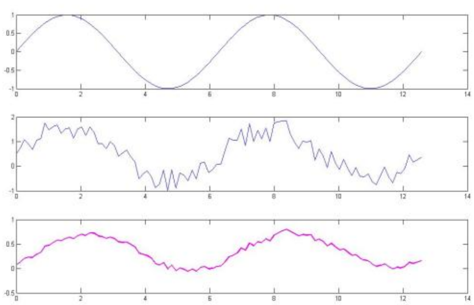
Vol. 1, Issue 3, June 2013

3.6.1 Response of Fraction Order and Integer Order Filter for Sine Wave input with Random Noise



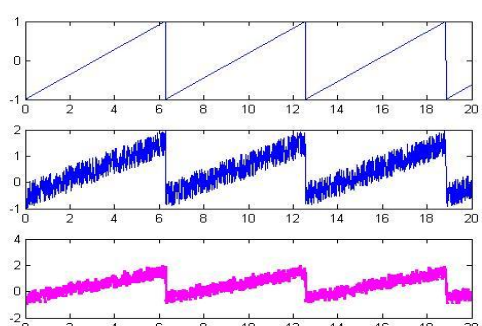
(a)



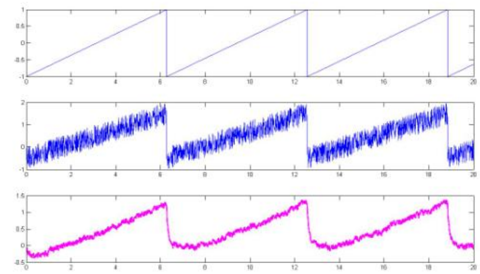
(b) α=0.55

Figure 8. (a) Performance of integer order filter and (b) performance of fractional order filter for sine wave with random noise as input of filter.

3.6.2 Response of Fraction Order and Integer Order Filter for Sawtooth Wave input with Random Noise



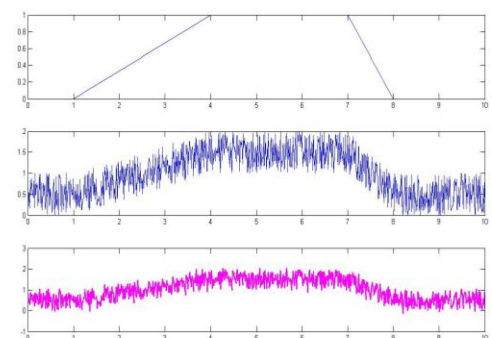
(a)



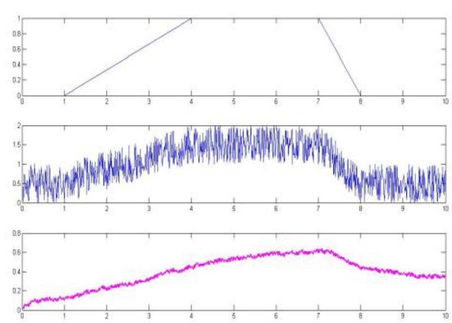
(b) α= 0.57

Figure 9. (a) Performance of integer order filter and (b) performance of fractional order filter for saw tooth wave with random noise as input of filter.

3.6.3.Response of Fraction Order and Integer Order Filter for input Trapezoidal Wave with Random Noise



(a)



(b) α =0.44

Figure10. (a) Performance of integer order filter and (b) performance of fractional order filter for trapezoidal wave with random noise as input of filter.

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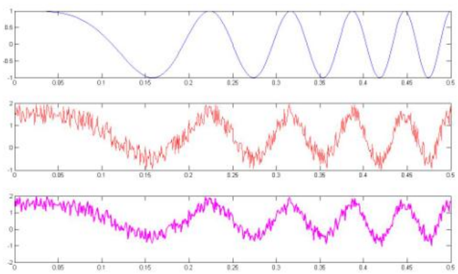
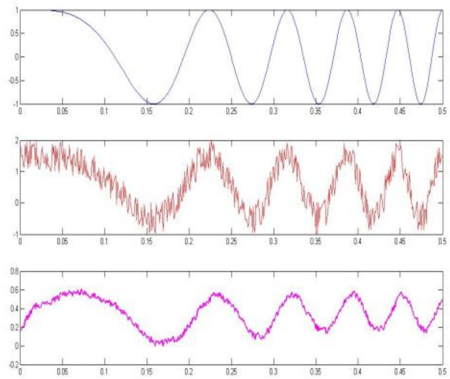
3.6.3. Response of Fraction Order and Integer Order Filter for input chirp signal with Random Noise



(a)



(b) α =0.47

Figure 11. (a) performance of integer order filter and (b) performance of fractional order filter for Chirp signal with random noise as input of filter.

IV. CONCLUSION

In this paper fractional order differential operator has been simulated in MATLAB for different Input signals and different value of α (fractional order). The simulated results show that the response of the system is noticeably different for the integer and non-integer values and it is observed that for gradual change of α from 0 to 1, the fractional order system gives the gradual change in the output response. Further fractional order filter is simulated in MATLAB for different input signals (signal with noise) and for different value of α (order of the operation), further compared with the resulted outputs of the integer order filter. So, it can be concluded that fractional order filter gives the better performance in comparison of integer order filter as noise is much suppressed in case of fractional order filter as in integer order filter . For better results optimum value of α (order of operation) is taken. So it can be concluded that output of a fractional order system will give different result if it is approximated by an integer order system. It is expected that the work will help the researchers to understand fractional order system behavior in a better way.

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